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Finite-temperature ϕ^4 theory from the 2PI effective action: Two-loop truncation

A. Arrizabalaga[1,a] and U. Reinosa[2]

[1] NIKHEF, Kruislaan 409, 1098 SJ, Amsterdam, The Netherlands
[2] Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany

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Abstract. Using resummation techniques based on the 2PI effective action we study the scalar ϕ^4 theory at finite temperature. We present an analytical as well as numerical study for a renormalized two-loop truncation of the action. Both the spectral properties and critical behaviour of the theory are investigated. Within the truncation, we explicitly check that the physical observables are UV-finite.

PACS. 11.10.Wx Finite-temperature field theory

1 Introduction

The thermodynamical properties of a physical system can be extracted from the effective potential, which is given as a function $\gamma[\phi]$ of a condensate field ϕ. In the thermodynamic limit, its stationary point defines the free energy $\mathcal{F}$ as

$$\mathcal{F} = \lim_{V\to\infty} \gamma[\bar{\phi}], \quad \text{with} \quad \frac{\delta\gamma[\phi]}{\delta\phi}\Big|_{\phi=\bar{\phi}} = 0. \qquad (1)$$

Once the free energy is known, all other thermodynamic quantities, such as the entropy or energy density, can be easily derived. At finite temperature collective phenomena are known to modify substantially the properties of the elementary excitations, so the use of a perturbative expansion around the free theory to calculate the effective potential becomes questionable and often leads to inconsistencies. One needs then to consider nonperturbative schemes. For situations where the effects of the collective phenomena can be conveniently captured by a suitable modification of the two-point functions, resummation schemes based on the 2PI effective action can be very useful [1,2]. The 2PI effective action consists of a reorganization of the perturbative expansion around dressed two-point functions, which are determined self-consistently for a given approximation/truncation. The technique is systematic and allows to go beyond mean-field and Hartree-type approximations, which are also included at lowest orders in the truncation. In this work we study the thermodynamics of the scalar ϕ^4 theory using a two-loop truncation of the 2PI effective action.

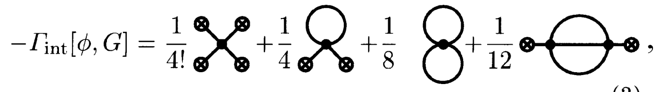
[a] e-mail: arrizaba@nikhef.nl

2 Two-loop truncation of the 2PI effective action

For the scalar ϕ^4 theory the 2PI effective action is usually parametrized as [3]

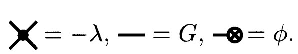
$$\Gamma_{\rm 2PI}[\phi,G] = \frac{1}{2}\phi\cdot G_0^{-1}\cdot\phi + \frac{1}{2}\,\mathrm{Tr}\left[\ln G^{-1} + \left(G_0^{-1} - G^{-1}\right)\cdot G\right] + \Gamma_{\rm int}[\phi,G], \qquad (2)$$

with ϕ and G generic one- and two-point functions and $A\cdot B$ a shorthand notation for the convolution of A and B. The term $\Gamma_{\rm int}$ contains the interactions and can be written as an expansion in terms of two-particle-irreducible (2PI) diagrams. Up to two-loops it is given by



(3)

where the Feynman rules are:



Within the approximation, "physical" one- and two-point functions $\bar{\phi}$ and $\bar{G}$ are determined self-consistenly as the stationary points of the 2PI effective action, *i.e.*

$$\frac{\delta\Gamma_{\rm 2PI}}{\delta G}\Big|_{\bar{G}[\phi],\phi} = 0 \quad \text{and} \quad \frac{\delta\Gamma_{\rm 2PI}}{\delta\phi}\Big|_{\bar{G}[\phi],\bar{\phi}} = 0. \qquad (4)$$

The stationary conditions (4) turn into a set of coupled implicit equations for $\bar{G}$ and $\bar{\phi}$ which have to be solved in order to calculate any physical quantity. In particular, the knowledge of the dressed two-point function $\bar{G}$ allows one to calculate the effective potential as $\gamma[\phi] = TV^{-1}\Gamma_{\rm 2PI}[\phi,\bar{G}[\phi]]$.

One of the main complications that arises when dealing with truncations of the 2PI effective action is the fact that two- and higher n-point functions are *not uniquely defined* [4]. In particular, a given truncation defines two possible two-point functions and three four-point functions. One of the two-point functions is given by the stationary value $\bar{G}$, while the other is related to the inverse curvature of the effective potential as $\hat{G}^{-1} = TV^{-1}[\delta^2\gamma[\phi]/\delta\phi^2]$. The three four-point functions and their corresponding vertices are given in terms of coupled Bethe-Salpeter–like equations [4]. Concerning *renormalization*, the ambiguity in the definition of the vertex functions implies that there are more than one counterterm of a given type [5]. A given counterterm is determined by applying a renormalization condition to the corresponding vertex. For consistency, identical renormalization conditions are applied to all counterterms of the same type. It is simpler to consider renormalization conditions applied at a reference temperature $T_\star$ for which the physical field configuration is $\bar{\phi}_\star = 0$. This requires that $T_\star > T_c$ if a critical temperature T_c exists. Defining the self-energy from Dyson's equation $\bar{\Sigma} = \bar{G}^{-1} - G_0^{-1}$, the renormalized equations for $\bar{\Sigma}$ and $\bar{\phi}$ are

$$\bar{\Sigma}(P) = \delta m_0^2 + \frac{\lambda+\delta\lambda_2}{2}\phi^2 + \frac{\lambda+\delta\lambda_0}{2}\int_K^T \bar{G}(K) - \frac{\lambda^2}{2}\phi^2\Theta(P) \tag{5}$$

and

$$\left(m^2+\delta m_2^2\right)\bar{\phi} = \frac{\lambda+\delta\lambda_4}{6}\bar{\phi}^3 + \frac{\lambda+\delta\lambda_2}{2}\bar{\phi}\int_K^T \bar{G}(K) - \frac{\lambda^2}{6}\bar{\phi}\int_K^T \bar{G}(K)\Theta(K), \tag{6}$$

with $\int_K^T$ a shorthand notation for the standard sum-integral over momentum K at temperature T, and

$$\Theta(P) = \int_K^T \bar{G}(K)\bar{G}(K+P). \tag{7}$$

The bar in ϕ has been omitted in the first equation to stress the fact that it can be solved for any value. This is needed, for instance, to calculate the effective potential $\gamma[\phi]$. The explicit expressions for the counterterms δm_0^2, δm_2^2, $\delta\lambda_0$, $\delta\lambda_2$ and $\delta\lambda_4$ can be found in ref. [4]. With those counterterms it can be shown that, for any value of T and ϕ, the results for $\bar{G}$, $\bar{\phi}$ and $\gamma[\phi]$ are UV-finite.

3 Numerical analysis

The gap and field equations (5) and (6) are solved numerically in Minkowski space. We split the self-energy into a local and a non-local part as $\bar{\Sigma}(P) = \bar{\Sigma}^l + \bar{\Sigma}^{nl}(P)$. The momentum-dependent part is expanded in both p_0 and $|\mathbf{p}|$ using N Chebyshev polynomials as

$$\bar{\Sigma}^{nl}(p_0,|\mathbf{p}|) = \sum_i^N\sum_j^N c_{ij}T_i(p_0)T_j(|\mathbf{p}|). \tag{8}$$

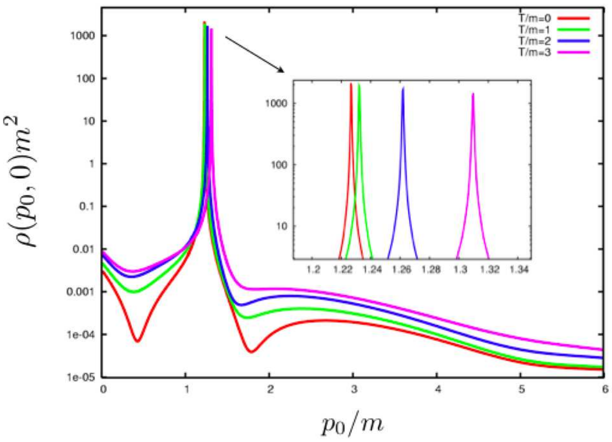


Fig. 1. Spectral function $\rho(p_0,0)$ for $\lambda = 1$, $\phi/m = 1$ and $T_\star/m = 1$ and several temperatures.

The solution to the gap and field equations is obtained by solving the matrix equation for the N^2 coefficients c_{ij} plus the local terms Σ^l and/or $\bar{\phi}^2$ by means of a multidimensional Newton-Raphson method. This numerical algorithm is fairly stable and converges after few iterations. For a given N, the numerical solution oscillates around the "full" solution, which is convenient for the calculation of integrated quantities, such as the effective potential. Unlike lattice-based methods, this technique also allows the use of large (small) UV (IR) cutoffs, which is useful for the study of *renormalization* and/or *critical phenomena*. Although numerically more expensive, the advantage of solving the equations in Minkowski space is that one can obtain directly spectral properties of the system without the need of analytic continuation. In particular, the spectral function $\rho(p_0,|\mathbf{p}|)$ can be constructed from the knowledge of the real and imaginary parts of the retarded self-energy, which come directly from solving eq. (5) (see fig. 1 for an example result at several temperatures). The width, the position of the quasiparticle pole and the effect of multiparticle contributions can be adequately studied with the presented algorithm.

We can also look at the critical behaviour in a system with broken symmetry. A simple quantity to compute is the critical temperature which can be read off the two-point functions at vanishing effective mass M. As we discussed previously there are two possible two-point functions ($\bar{G}$ and $\hat{G}$), for which the corresponding effective masses are (in the symmetric phase)

$$M(T,\lambda)^2 = m^2 + \Sigma^l(T,\lambda), \tag{9}$$

$$\hat{M}(T,\lambda)^2 = \hat{G}^{-1}(p=0,T,\lambda) = \frac{\delta^2\Gamma[\phi,\bar{G}[\phi]]}{\delta\phi^2}\Big|_{\bar{\phi}}(p=0,T,\lambda). \tag{10}$$

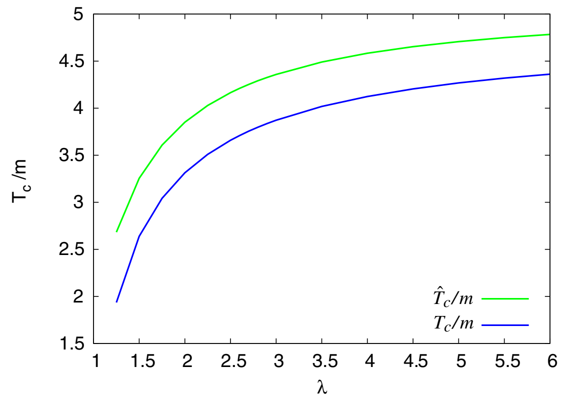


Fig. 2. Critical temperatures for several couplings ($T_\star/m = 5$).

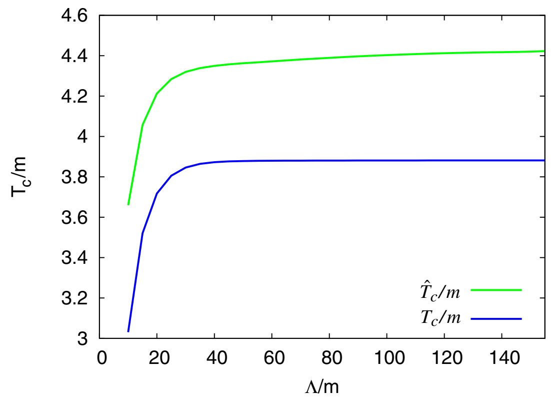


Fig. 3. UV cutoff dependence for $\lambda = 3$ ($T_\star/m = 5$).

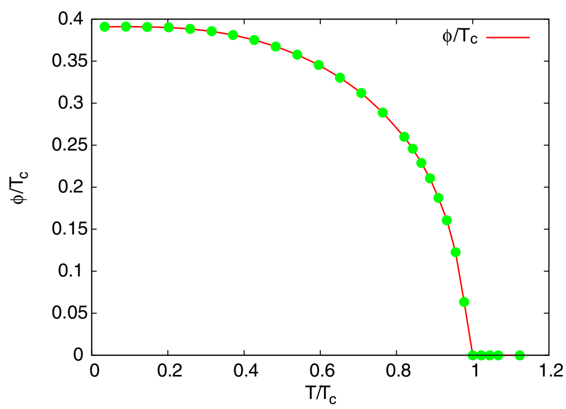


Fig. 4. Field expectation value $\bar{\phi}^2$ as a function of T for $\lambda = 3$ and $T_\star/m = 5$.

Starting from $T_\star$ and reducing the temperature, the critical values T_c and $\hat{T}_c$ are found when M^2 and $\hat{M}^2$ vanish. The results for several couplings are shown in fig. 2. We can also vary the UV cutoff in the calculation of the critical temperatures and therefore check the validity of the renormalization procedure (see fig. 3). We find that both quadratic and logarithmic divergences (present, respectively, in T_c and $\hat{T}_c$) are properly renormalized and hence the renormalization is satisfactory.

Solving in addition the field equation (6) allows us to study the phase transition. In fig. 4 we show the behavior

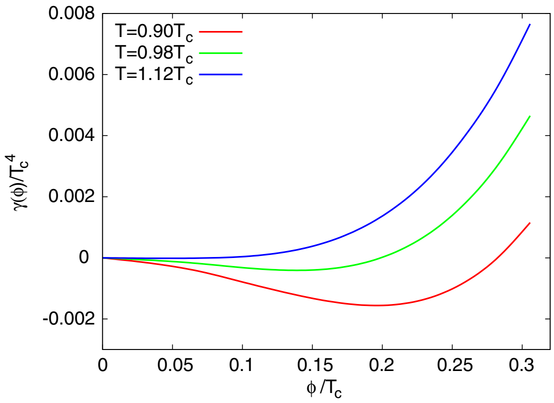


Fig. 5. Effective potential for $\lambda = 3$ and $T_\star/m = 5$.

of the order parameter $\bar{\phi}$ as a function of the temperature. We see that the transition is clearly of second order. This agrees with lattice studies as well as results from other non-perturbative approaches [6]. So, unlike mean-field or Hartree approximations, which predict the wrong order for the transition, methods based on the 2PI effective action give the correct result. A similar conclusion is reached by computing directly the effective potential $\gamma[\phi]$ (see fig. 5). Finally, we checked that $\gamma[\bar{\phi}]$, from which a renormalized free energy and pressure can be extracted, is indeed cutoff independent.

4 Conclusions and prospects

We have shown for a simple scalar ϕ^4 theory that the resummation methods based on the 2PI effective actions can be used to obtain physically meaningful (*i.e.*, renormalized) information about the thermodynamics, spectral properties and phase structure of a given system beyond mean-field or Hartree approximations. The natural next step is to apply the technique to theories physically more relevant, specially to those for which most non-perturbative finite-temperature studies are limited to mean-field and/or Hartree-type approximations. Of particular importance to heavy-ion phenomenology are chiral effective theories such as sigma or Nambu-Jona-Lasinio–type models (some work in that direction is underway [2]) and gauge theories[1].

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